

Dual symmetry in a generalized Maxwell theory

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(Dated: October 11, 2016)

We examine Podolsky's electrodynamics, which is noninvariant under the usual duality transformation. We deduce a generalization of Hodge's star duality, which leads to a dual gauge field and restores to a certain extent the dual symmetry. The model becomes fully dual symmetric asymptotically, when it reduces to the Maxwell theory. We argue that this strict dual symmetry directly implies the existence of the basic invariants of the electromagnetic fields.

PACS numbers: 11.10.-z, 11.15.-q

Keywords: Gauge field theories, Symmetry and conservation laws

I. INTRODUCTION

Many symmetries that led to our understanding of quantum field theories, have their origin in the Maxwell theory describing the electromagnetic interactions. One such symmetry is duality which, in its simplest form, is the invariance of the source-free Maxwell equations under the interchange of electric and magnetic fields. Although the physical meaning of the dual symmetry is not yet entirely clear, there has been much work on the implications of this symmetry [1]. Apart from the related duality in supersymmetric and superstring theories [2–5], there has also been some interest in the extension of the dual symmetry to the classical Yang-Mills theory [6]. In this non-Abelian gauge theory, the Hodge dual field $*F_{\alpha\beta}$, in contrast to the gauge field $F_{\alpha\beta}$, is not derivable from a potential, so that the usual dual symmetry is in general absent.

The main purpose of this work is to study this problem in the context of a simpler (Abelian) gauge theory, where the issues associated with the lack of a manifest dual symmetry may be easier to analyze and understand. One such model is Podolsky's generalized Maxwell theory [7], which preserves the linear character of the field equations, yet avoids the divergences present in the Maxwell theory at short distances. Due to this behaviour, certain difficulties of classical electromagnetism, like the 4/3 problem or non-causal effects are naturally solved in the context of Podolsky's electrodynamics [8, 9]. At large distances compared with some quantum scale, this model effectively reduces to the Maxwell theory. (For more recent studies of this theory see, for example, references [10–15].)

In Podolsky's model, a strict dual symmetry is also absent, which is apparent in the fact that the usual Hodge dual field is not generally a gauge field. We derive a generalized Hodge star duality, which reduces to the usual star operation at large distances. This leads to a dual gauge field derivable from a potential and restores, to a certain extent, the dual symmetry. This behaviour may reproduce, in a simpler context, some analogous features which occur in the Yang-Mills theory. Thus, the present analysis may be helpful to understand a possible mechanism for the restoration of a dual symmetry in non-Abelian gauge theories.

In a source-free region, Podolsky electrodynamics exhibits asymptotically, at large distances compared with the electron Compton wavelength, a complete dual symmetry. Since in this case the model practically reduces to the Maxwell theory, it may be worth to examine some implications of the strict dual symmetry. We thereby deduce the electromagnetic invariants of the source-free Maxwell equations, without an explicit resort to the Lorentz transformations properties of the fields. Using just the linearity and locality of the transformations, we show that the full duality directly implies the existence of the Lorentz invariants $E^2 - B^2$ and $\vec{E} \cdot \vec{B}$.

II. DUALITY IN PODOLSKY ELECTRODYNAMICS

This theory [7] may be described, in the source free case, by the action

$$S(l) = -\frac{1}{16\pi} \int d^4x \left[F^{\alpha\beta} F_{\alpha\beta} + 2l^2 (\partial_\beta F^{\alpha\beta}) (\partial^\gamma F_{\alpha\gamma}) \right], \quad (1)$$

where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the electromagnetic gauge field tensor and l is a length scale induced by quantum processes, which regularises the classical Maxwell theory at short distances [8, 9].

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The Euler-Lagrange equations following from (1) are given by

$$(1 - l^2 \square) \partial^\alpha F_{\alpha\beta} = 0 \quad (2)$$

which leads to the generalized Gauss and Ampère equations

$$(1 - l^2 \square) \nabla \cdot \vec{E} = 0; \quad (1 - l^2 \square) \left[\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \right] = 0. \quad (3)$$

By virtue of the above form of the electromagnetic field tensor $F_{\alpha\beta}$, the Hodge dual tensor

$$*F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} F^{\mu\nu} \quad (4)$$

satisfies the Bianchi identity

$$\partial^\alpha *F_{\alpha\beta} = 0. \quad (5)$$

Since $*F_{\alpha\beta}$ can be obtained from $F_{\alpha\beta}$ by interchanging $\vec{E} \rightarrow \vec{B}$ and $\vec{B} \rightarrow -\vec{E}$, (5) is equivalent to the homogeneous equations

$$\nabla \cdot \vec{B} = 0; \quad \left[\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} \right] = 0. \quad (6)$$

One can see from (3) and (6) that Podolsky's theory is no longer dual invariant in general. We will next argue that there exists some potential \underline{A}_μ , which satisfies the equation

$$(1 - l^2 \square) *F_{\alpha\beta} = \partial_\alpha \underline{A}_\beta - \partial_\beta \underline{A}_\alpha \quad (7)$$

so that (2) may be interpreted as the Bianchi identity for $(1 - l^2 \square) *F_{\alpha\beta}$. To this end, we note that (7) implies the relation

$$(1 - l^2 \square) [\partial_\rho *F_{\alpha\beta} + \partial_\alpha *F_{\beta\rho} + \partial_\beta *F_{\rho\alpha}] = 0. \quad (8)$$

Due to the fact that the dual operation is reflexive, $*(F_{\alpha\beta}) = -F_{\alpha\beta}$, (8) leads to the Eq. (2):

$$-(1 - l^2 \square) \partial^\alpha *(F_{\alpha\beta}) = (1 - l^2 \square) \partial^\alpha F_{\alpha\beta} = 0. \quad (9)$$

In order to determine explicitly the potential \underline{A}_μ , we use an alternative method [6], where $F_{\alpha\beta}$ are considered as field variables, subject to the constraint (5). By imposing this condition, one can remove the inherent redundancies comprised in the set of variables $\{F_{\alpha\beta}\}$. This procedure can be enforced by introducing Lagrange multipliers Λ_μ and constructing the action

$$\begin{aligned} \bar{S}(l) = & -\frac{1}{16\pi} \int d^4x [F^{\alpha\beta} F_{\alpha\beta} \\ & + 2l^2 (\partial_\beta F^{\alpha\beta}) (\partial^\gamma F_{\alpha\gamma}) - 4\Lambda^\beta \partial^\alpha *F_{\alpha\beta}], \end{aligned} \quad (10)$$

Extremizing (10) with respect to $F_{\alpha\beta}$ and Λ^β we then get, apart from (5), the equation

$$\begin{aligned} F_{\alpha\beta} - l^2 \partial^\rho (\partial_\beta F_{\alpha\rho} - \partial_\alpha F_{\beta\rho}) = \\ -\frac{1}{2} \epsilon_{\alpha\beta\mu\nu} (\partial^\mu \Lambda^\nu - \partial^\nu \Lambda^\mu). \end{aligned} \quad (11)$$

Expressing the electromagnetic tensors in terms of their duals, we obtain from (11), after some algebra which makes use of the constraint (5), the relation

$$(1 - l^2 \square) *F_{\alpha\beta} = \partial_\alpha \Lambda_\beta - \partial_\beta \Lambda_\alpha. \quad (12)$$

Comparing (7) and (12), we see that the potential \underline{A}_μ is just the Lagrange multiplier Λ_μ .

We note here that the general solution of (7) can be written in the form

$$*F_{\alpha\beta} = \mathcal{F}_{\alpha\beta} + (1 - l^2 \square)^{-1} (\partial_\alpha \underline{A}_\beta - \partial_\beta \underline{A}_\alpha) \quad (13)$$

where the operator $(1 - l^2 \square)^{-1}$ may be interpreted, for slowly varying fields, as the series $1 + l^2 \square + \dots$ and $\mathcal{F}_{\alpha\beta}$ is a solution of the homogeneous equation

$$(1 - l^2 \square) \mathcal{F}_{\alpha\beta}(x) = 0. \quad (14)$$

It turns out that there exist non-trivial solutions of this equation. For example, in the static case one finds that

$$\mathcal{F}_{\alpha\beta}(\vec{x}) = \nabla^2 \int d^3x' \frac{J_{\alpha\beta}(\vec{x}')}{|\vec{x} - \vec{x}'|} \exp\left(-\frac{|\vec{x} - \vec{x}'|}{l}\right) \quad (15)$$

where $J_{\alpha\beta}$ is an external source which vanishes at points \vec{x} inside the source free region. We can see from (15), that $\mathcal{F}_{\alpha\beta}(\vec{x})$ is exponentially vanishing at distances much larger than l , or equivalently in the limit $l \rightarrow 0$, as expected from (14).

From the above equations, it follows that, in contrast to the gauge field $F_{\alpha\beta} = \partial_\alpha \underline{A}_\beta - \partial_\beta \underline{A}_\alpha$, the dual field $*F_{\alpha\beta}$ is generally not derivable from a potential. However, we can see from (7) that one may generalize the Hodge star duality as

$$\underline{F}_{\alpha\beta} \equiv (1 - l^2 \square) *F_{\alpha\beta} = \frac{1}{2} \epsilon_{\alpha\beta\mu\nu} (1 - l^2 \square) F^{\mu\nu}, \quad (16)$$

so that the modified dual field $\underline{F}_{\alpha\beta}$ is also a gauge field derivable from the potential \underline{A}_μ : $\underline{F}_{\alpha\beta} = \partial_\alpha \underline{A}_\beta - \partial_\beta \underline{A}_\alpha$. Such a field satisfies, from (5), the equation

$$(1 - l^2 \square)^{-1} \partial^\alpha \underline{F}_{\alpha\beta} = 0. \quad (17)$$

It may be verified that the solutions of equations (2) and (17) are interchangeable, up to terms of order $\exp(-|\vec{x}|/l)$ which become small at distances larger than l . Thus, the dual gauge field $\underline{F}_{\alpha\beta}$ restores to this extent the dual symmetry. A complete dual symmetry is obtained in the asymptotic region where the parameter l may be effectively set equal to zero, in which case Podolsky electrodynamics reduces to the Maxwell theory.

III. LORENTZ INVARIANTS FROM DUAL SYMMETRY

We will show next that the full dual symmetry of the source-free Maxwell equations directly leads to the existence of the electromagnetic invariants. We recall that in

a relativistic covariant theory, the electric and magnetic fields transform as a Lorentz tensor, so that the electromagnetic fields in two inertial frames K' and K satisfy the conditions [16, 17]

$$E'^2 - B'^2 = E^2 - B^2 \quad \text{and} \quad \vec{E}' \cdot \vec{B}' = \vec{E} \cdot \vec{B}. \quad (18)$$

In order to derive these invariants from duality, let us consider a general transformation which relates the fields in K' and K . In a homogeneous and isotropic space-time, which is assumed by the relativity principle, the transformation should be linear. Moreover, such a transformation cannot contain derivatives of the fields, since it must be local. The locality ensures that the values of the fields at some space-time point in K , uniquely determine the fields observed in K' at the same space-time point. Thus, a general transformation which relates \vec{E}' to \vec{E} and \vec{B} must have the form

$$\begin{aligned} \vec{E}' = & a_1 \vec{E} + a'_1 \vec{B} + b_1 \vec{\beta} \times \vec{B} + b'_1 \vec{\beta} \times \vec{E} \\ & + d_1 (\vec{\beta} \cdot \vec{E}) \vec{\beta} + d'_1 (\vec{\beta} \cdot \vec{B}) \vec{\beta} \end{aligned} \quad (19)$$

where $c\vec{\beta}$ is the velocity of K' relative to K and the dimensionless coefficients $a_1, a'_1, b_1, b'_1, d_1, d'_1$ are real functions of β^2 . The six structures in (19) may be independent, since transformations in 4-dimensional space-time may be characterized by six parameters, corresponding for example to three rotation angles and three Lorentz boosts. Using the dual symmetry under $\vec{E} \rightarrow \vec{B}$ ($E' \rightarrow B'$) and $\vec{B} \rightarrow -\vec{E}$ ($B' \rightarrow -E'$), it follows from (19) that the corresponding transformation of the magnetic field should have the form

$$\begin{aligned} \vec{B}' = & a_1 \vec{B} - a'_1 \vec{E} - b_1 \vec{\beta} \times \vec{E} + b'_1 \vec{\beta} \times \vec{B} \\ & + d_1 (\vec{\beta} \cdot \vec{B}) \vec{\beta} - d'_1 (\vec{\beta} \cdot \vec{E}) \vec{\beta} \end{aligned} \quad (20)$$

It is now convenient to introduce the complex electromagnetic vector field [1, 16]

$$\vec{F} = \vec{E} + i\vec{B}. \quad (21)$$

Then, it follows from (19) and (20) that the general linear transformation of the complex vector field \vec{F} can be written as

$$\vec{F}' = a\vec{F} - ib\vec{\beta} \times \vec{F} + d(\vec{\beta} \cdot \vec{F})\vec{\beta}, \quad (22)$$

where $a = a_1 - ia'_1$, $b = b_1 + ib'_1$ and $d = d_1 - id'_1$ are complex coefficients. The inverse transformation can be obtained by changing $\vec{\beta} \rightarrow -\vec{\beta}$ in (22):

$$\vec{F} = a\vec{F}' + ib\vec{\beta} \times \vec{F}' + d(\vec{\beta} \cdot \vec{F}')\vec{\beta}. \quad (23)$$

Substituting (23) in (22), we obtain the consistency conditions

$$a^2 - b^2\beta^2 = 1; \quad a + d\beta^2 = 1, \quad (24)$$

where we used the fact that $a \rightarrow 1$ as $\beta \rightarrow 0$.

Let us consider the parallel and perpendicular components of (22) with respect to $\vec{\beta}$. Using (24) we then obtain the relations

$$\vec{F}'_{\parallel} = (a + d\beta^2)\vec{F}_{\parallel} = \vec{F}_{\parallel} \quad (25a)$$

$$\vec{F}'_{\perp} = a\vec{F}_{\perp} - ib\vec{\beta} \times \vec{F}_{\perp}. \quad (25b)$$

These correspond to a rotation around $\vec{\beta}$ by a complex angle $i\theta$ where $\theta = \tanh^{-1}(\beta b/a)$. Thus, we get

$$\vec{F}'_{\perp} \cdot \vec{F}'_{\perp} = (a^2 - b^2\beta^2)\vec{F}_{\perp} \cdot \vec{F}_{\perp} = \vec{F}_{\perp} \cdot \vec{F}_{\perp}. \quad (26)$$

From (25a) and (26) it follows that $\vec{F}' \cdot \vec{F}' = \vec{F} \cdot \vec{F}$ and hence

$$E'^2 - B'^2 + 2i\vec{E}' \cdot \vec{B}' = E^2 - B^2 + 2i\vec{E} \cdot \vec{B} \quad (27)$$

which leads to the Lorentz invariants (18). We have thus shown that the electromagnetic invariants can be deduced from the dual symmetry of the source-free Maxwell equations.

We finally remark that a photon mass would break the dual symmetry of the Maxwell equations. Such a mass may be generated, for example, by the Stueckelberg mechanism [18, 19] which preserves gauge invariance and renormalizability. Hence, the fact that the photon remains massless could be explained if duality would hold at a fundamental level. (Measurements of the galactic vector potential place an upper bound on the photon mass of 10^{-18} eV/c² [20].) This is an open problem which might be worthy of further examination.

ACKNOWLEDGMENTS

We thank CNPq (Brazil) for a grant and Prof. J. C. Taylor for helpful discussions.

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